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ORIGINAL ARTICLE

An efficient procedure for dynamic lot-sizing model with demand time windows

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Abstract We consider a dynamic lot-sizing model with demand time windows where n demands need to be scheduled in T production periods. For the case of backlogging allowed, an $O(T^3)$ algorithm exists under the non-speculative cost structure. For the same model with somewhat general cost structure, we propose an efficient algorithm with $O(\max\{T^2, nT\})$ time complexity.

Keywords Dynamic lot-sizing model · Demand time window · Non-speculative cost structure · Dynamic programming · Production

1 Introduction

In past mass production systems where the relationship between suppliers and customers was not a particularly important factor, demands were generated by aggregating requirements by periods based on forecasting. That is, each demand was given by a period. In this present economy, however, as strategic partnership is a major success factor for both suppliers and customers, demands are made in long-term and volume-based contracts. As a result, demands are given by an interval of periods called *time windows* for the long-term agreement and no penalty is incurred if the total required volume is met during the periods. Such a situation of demand time windows can also be seen in third party logistics and vendor managed inventory practices (Lee et al. 2001; Jaruphongsa et al. 2004 a,b).

In this paper, we consider the single-item dynamic lot-sizing model with demand time windows where n demands need to be scheduled in T production periods. Each demand's time window is specified by the earliest due date (EDD) and the latest due date (LDD), during which no inventory and backlogging costs are incurred.

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When each demand's time window is just a single period (EDD = LDD), the model is reduced to the *classical* dynamic lot-sizing model. Since the seminal work of Wagner and Whitin (1958), extensive research has been done for the classical dynamic lot-sizing model (Wolsey 1995; Brahimi et al. 2006). In general, however, not much study has been conducted on the model with demand time windows. To deal with the time window consideration in production planning, Lee et al. (2001) first studied the dynamic lot-sizing model with time windows. To generate optimal production schedules, they proposed two procedures: one with complexity $O(T^2)$ for the case with 'no backlogging' and the other with complexity $O(T^3)$ for the case with 'backlogging'. The procedures are developed under the *non-speculative* cost structure that the unit production cost in period t is at most the unit production cost in period $t-1$ and also at most the unit production cost plus the backlogging cost in period $t+1$. For the general cost structure where no such restrictions are imposed, Hwang and Jaruphongsa (2006) recently provided an optimal procedure with complexity $O(nT^3)$.

The classical dynamic lot-sizing model under the non-speculative cost structure can be solved easily in $O(T^2)$ by ordinary dynamic programming procedures. Furthermore, by applying the monotonicity property intrinsic in the model (or the Monge property in general; Aggarwal and Park 1993), we can increase the efficiency of the main recursion procedure of the dynamic programming once we have computed the necessary cost data by preprocessing. Federgruen and Tzur (1991), Wagelmans et al. (1992) and Aggarwal and Park (1993) independently demonstrated the remarkably fast locating of an optimal production schedule in a time of $O(T)$. Therefore, for the dynamic lot-sizing model with time windows it would be worthwhile trying to develop a more efficient algorithm with less complexity than that presented by Lee et al. (2001). As stated in Lee et al. (2001), when backlogging is not allowed the development of a more efficient algorithm with complexity less than $O(T^2)$ seems very unlikely. Therefore, in this study we focus on the case with backlogging allowed.

In this paper, we present a solution procedure with complexity $O(\max\{T^2, nT\})$ for the dynamic lot-sizing model with time windows under non-speculative cost structure where backlogging is allowed. In our case, we have a rather more general cost structure than the non-speculative cost structure that Lee et al. (2001) were restricted to; it is merely assumed that the unit production cost in period t is at most the unit production cost in period $t-1$ without any restriction on backlogging costs. Note that the number of demands n is $O(T^2)$ since we aggregate and designate demands by intervals of periods. Hence, when n increases to the maximum number of time windows in T production periods, the procedure in this paper has the same complexity of $O(T^3)$ as the procedure by Lee et al. (2001). However, the complexity $O(\max\{T^2, nT\})$ seems the best possible result achievable in theory since the preprocessing itself could not be carried out in time less than $O(\max\{T^2, nT\})$. As we shall see, the complexity of the main recursion procedure is merely $O(T^2)$.

In the next section, we first define the analytical model for the problem and review optimality properties developed so far. In Sect. 3, the optimal solution procedure is presented under the assumption that all the necessary cost data has been calculated by preprocessing. In order to increase the preprocessing efficiency, the concept of selection measures and their properties are introduced in Sect. 4. The preprocessing steps are described in Sect. 5, which make the optimal solution procedure run up to $O(\max\{T^2, nT\})$. Finally, we present the conclusion.

2 The model and review of the optimality properties

Suppose that we have n demands to be satisfied over the planning horizon T. We define the following parameters and decision variables for our model.

Parameters

- d_i : the required quantity for demand i for $i = 1, \ldots, n$.
- $[E_i, L_i]$: the time window of demand i for $i = 1, \ldots, n$ where demand is EDD and LDD are denoted by E_i and L_i, respectively. If a demand is satisfied during its time window no costs of inventory holding and backlogging are incurred.
- K_t: the fixed cost of production in period t.
- p_t : the unit production cost in period t.
- h_t : the unit holding cost in period t.
- g_t : the unit backlogging cost in period t.

For the purpose of notational convenience, we additionally give definitions for period

$$T+1: \text{ let } K_{T+1} = p_{T+1} = h_{T+1} = g_{T+1} = \infty.$$

Decision variables

- y_{it} : the amount dispatched in period t for demand i for $i = 1, \ldots, n$ and $t = 1, \ldots, T$.
- x_t : the amount replenished in period t for $t = 1, \ldots, T$.
- I_t^+ : the inventory level in period t for $t = 1, \ldots, T$.
- I_t^- : the backlogging level in period t for $t = 1, \ldots, T$.

The mathematical formulation of the problem is given by:

$$\begin{aligned}
&\text{Minimize} \sum_{t=1}^{T} \left(K_t\delta_t + p_t x_t + h_t I_t^+ + g_t I_t^- \right) \\
&\text{Subject to} \\
&x_t + (I_{t-1}^+ - I_{t-1}^-) - \sum_{i=1}^{n} y_{it} = (I_t^+ - I_t^-), && t = 1, \ldots, T, \\
&x_t \le M\delta_t, && t = 1, \ldots, T, \\
&\sum_{t \in [E_i, L_i]} y_{it} = d_i, && i = 1, \ldots, n, \\
&y_{it} \ge 0, && i = 1, \ldots, n, \quad t \in [E_i, L_i], \\
&y_{it} = 0, && i = 1, \ldots, n, \quad t \notin [E_i, L_i], \\
&x_t \ge 0, \quad I_t^+ \ge 0, \quad I_t^- \ge 0, \quad \delta_t \in \{0, 1\}, && t = 1, \ldots, T, \\
&I_0^+ = I_0^- = I_T^+ = I_T^- = 0,
\end{aligned}$$

where M is a very large number.

Lee et al. (2001) assumed the non-speculative cost structure that $p_{t-1} \ge p_t$ for all $t = 2, 3, \ldots, T$ and that $p_{t+1} + g_{t+1} \ge p_t$ for all $t = 1, 2, \ldots, T-1$. However, in this paper we consider the more general case that $p_{t-1} \ge p_t$ for all $t = 2, 3, \ldots, T$ with no restriction on backlogging cost.

We next summarize the two optimality properties developed in Lee et al. (2001). It is not hard to see that the properties in the following hold under our cost structure of $p_{t-1} \ge p_t$ as well as under the non-speculative cost structure.

Property 1 *There is an optimal solution such that* $y_{it} = d_i$ *for some* $t \in [E_i, L_i]$ *for* $i = 1, \ldots, n$.

That is, there exists an optimal solution such that each demand i is satisfied by a single dispatch.

Property 2 *It is optimal to satisfy a demand either by the last replenishment before its LDD or by the first replenishment at or after its LDD.*

Hence, if a demand has a production in its LDD, then it is not replenished in any period earlier than the LDD.

Based on these two properties, an algorithm with complexity $O(T^3)$ was developed by Lee et al. (2001). The complexity of $O(T^3)$ results from the preprocessing steps to find cost data since the main dynamic programming procedure is executed in less time of $O(T^2)$. In our algorithm an optimal production schedule will be found in merely time $O(\max\{T^2, nT\})$. This is accomplished by reducing the computational steps in preprocessing. The time reduction from $O(T^3)$ to $O(\max\{T^2, nT\})$ is chiefly based on the following *order invariance* property: when satisfying a demand by inventory from a production at period λ, its unit cost is the unit production cost at λ plus the unit inventory holding costs accrued. In the case that the same demand is fulfilled by backlogging from a production at period γ, where $\gamma > \lambda$, the unit cost for satisfying the demand is the unit production cost at γ plus the unit backlogging cost accrued. We sort demands in nondecreasing (or nonincreasing) order by the difference between these two unit costs and suppose that demand i precedes demand j in the sorted list. Then, we have the invariant order of demands in the sense that demand i always precedes demand j for any two production periods λ and γ, $1 \le \lambda < \gamma \le T$, whenever time windows of both demands i and j belong to the interval $[\lambda, \gamma]$.

To denote cumulative inventory holding cost and cumulative backlogging cost, we define $h_{s,t}$ and $g_{s,t}$. For any $1 \le s \le t \le T$, let

$$h_{s,t} = h_s + h_{s+1} + \cdots + h_t \quad \text{and} \quad g_{s,t} = g_s + g_{s+1} + \cdots + g_t.$$

We also let $h_{s,t} = g_{s,t} = 0$ if $s > t$.

Now, we present the following example to illustrate the optimality properties and to help the reader understand notations that will be developed in the next section.

Example 1 We are going to satisfy eight demands $1, 2, \ldots, 8$ in the periods $1, 2, \ldots, 14$. As shown in Table 1, most unit costs of production, inventory holding and backlogging are 1 except that the unit production costs of periods 1, 2, and 3 are all 3, the unit holding cost of period 5 is 3, and the unit backlogging costs of periods 8 and 10 are 2 and 4, respectively. The time windows of the eight demands are specified in Table 2. In addition, in Fig. 1, each demand's time window is denoted by a line segment across the 14 periods and its index is represented in the circle. Suppose that we have two consecutive productions in periods $\lambda = 3$ and $\gamma = 12$. Then, from Property 2 we can see that demands 6, 7, and 8 are replenished at or after the period γ. However, it is not certain whether the other demands are replenished either from period λ or γ since the decision relies on production costs with inventory holding and backlogging costs. Each demand's unit replenishment costs from periods λ and γ, which include holding and backlogging costs, respectively, are described in the left and right of its time window, respectively. With the example of demand 3, the unit production cost at $\lambda = 3$ plus the cumulative inventory cost is $p_\lambda + h_{\lambda, E_3 - 1} = 4$ and the unit production

Table 1 Unit cost data in Example 1

Periods	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Unit production cost	3	3	3	1	1	1	1	1	1	1	1	1	1	1
Unit holding cost	1	1	1	1	3	1	1	1	1	1	1	1	1	1
Unit backlogging cost	1	1	1	1	1	1	1	2	1	4	1	1	1	1

Table 2 Demand time windows in Example 1

Demands	1	2	3	4	5	6	7	8
Time windows	[1,4]	[1,11]	[4,8]	[6,9]	[5,10]	[9,12]	[2,14]	[11,14]

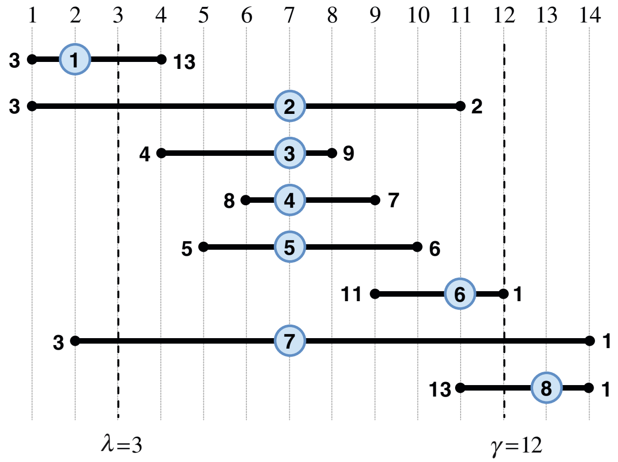

Fig. 1 Demands of Example 1 with time windows and unit replenishment costs

cost at $\gamma = 12$ plus the cumulative backlogging cost is $p_\gamma + g_{L_3,\gamma-1} = 9$. In the case of demands 1 and 2, no holding cost but the production cost of $p_\lambda = 3$ is incurred. Comparing the costs described in the figure between replenishments in λ and γ, we can see that demands 1, 3, and 5 are replenished from λ, and 2 and 4 from γ.

From this to end, most arguments are based on intervals of periods. By the interval $[\lambda, \gamma]$ we assume that there are consecutive productions at the two periods λ and $\gamma, 1 \le \lambda < \gamma \le T + 1$. We say that a demand i is *proper* with respect to the interval $[\lambda, \gamma]$ if $\lambda < E_i \le L_i < \gamma$. Also, any demand i with $E_i \le \lambda \le L_i < \gamma$ is called λ-*crossing* with respect to the interval $[\lambda, \gamma]$. Then, if a demand is either proper or λ-crossing with respect to the interval $[\lambda, \gamma]$, it is called *demand with* the interval $[\lambda, \gamma]$. In Example 1, the demands with $[\lambda, \gamma]$ are 1 through 5 for which the λ-crossing demands are 1 and 2, and the proper demands are 3, 4, and 5.

For notational convenience, we introduce a simple indicator function f defined as $f(s, t) = 1$ if $s \le t$, 0 otherwise. It can be used for checking whether a demand is proper or crossing with respect to an interval. By the multiplication of two values $f(\lambda, E_i - 1)$ and $f(L_i, \gamma - 1)$, we can determine whether a demand i is proper with the interval $[\lambda, \gamma]$ or not. The multiplication $f(\lambda, E_i - 1) \cdot f(L_i, \gamma - 1)$ results in one if and only if demand i properly belongs to the interval. Similarly, for checking λ-crossing of a demand i, we can use the value of the multiplication $f(E_i, \lambda) \cdot f(\lambda, L_i) \cdot f(L_i, \gamma - 1)$. The value is one if and only if the demand is λ-crossing with respect to the interval $[\lambda, \gamma]$.

3 Optimal solution procedure

Before describing the optimal solution procedure, we first present the necessary notations, which make it possible to arrange and group demands by EDD and LDD.

- $\alpha(i)$: the sorted list of demands in nondecreasing order of EDD so that $E_{\alpha(i)} \leq E_{\alpha(i+1)}$ for $i = 1, 2, \ldots, n-1$.
- $a(t)$: the largest index i in the list α such that its corresponding demand $\alpha(i)$ has EDD equal to t, i.e., $E_{\alpha(i)} = t$ for $t = 1, 2, \ldots, T$. If no such $a(t)$ exists, we let $a(t) = a(t-1)$, where $a(0) = 0$. Then all the demands with EDD of t are the ones $\alpha(i)$ for $a(t-1) < i \leq a(t)$.
- $\beta(i)$: the sorted list of demands in nondecreasing order of LDD so that $L_{\beta(i)} \leq L_{\beta(i+1)}$ for $i = 1, 2, \ldots, n-1$.
- $b(t)$: the largest index i in the list β such that its corresponding demand $\beta(i)$ has LDD equal to t, i.e., $L_{\beta(i)} = t$ for $t = 1, 2, \ldots, T$. If no such $b(t)$ exists, we let $b(t) = b(t-1)$, where $b(0) = 0$. Then all the demands with LDD of t are the ones $\beta(i)$, for $b(t-1) < i \leq b(t)$.

Note that both the lists $\alpha(i)$ and $\beta(i)$ can be computed in $O(n \log n)$. Also, we can have $a(t)$ and $b(t)$ for $t = 1, 2, \ldots, T$ in $O(n + T)$ using the lists $\alpha(i)$ and $\beta(i)$, respectively.

We next consider the associated cost data when we have consecutive productions at two periods λ and γ.

- $\text{hload}(\lambda, \gamma)$: the total amount of the proper demands with the interval $[\lambda, \gamma]$ which are replenished from period λ by carrying inventory.
- $\text{hcost}(\lambda, \gamma)$: the inventory holding cost (not including production cost) for satisfying the proper demands corresponding to $\text{hload}(\lambda, \gamma)$.
- $\text{phload}(\lambda, \gamma)$: the total amount of the demands with the interval $[\lambda, \gamma]$ which are replenished from period λ. We note that $\text{phload}(\lambda, \gamma) \geq \text{hload}(\lambda, \gamma)$.
- $\text{pgcost}(\lambda, \gamma)$: the total production (not including setup) and backlogging costs in period γ for satisfying the demands with $[\lambda, \gamma]$ which are cheaper than replenishment by period λ.

In the previous example, we have $\text{hload}(\lambda, \gamma) = d_3 + d_5$, $\text{hcost}(\lambda, \gamma) = d_3 + 2d_5$, $\text{phload}(\lambda, \gamma) = d_1 + d_3 + d_5$, and $\text{pgcost}(\lambda, \gamma) = 2d_2 + 7d_4$.

In the classical lot-sizing model, most of the decomposition principle allowing dynamic programming is based on *regeneration* periods t which do not carry or backlog any inventory. However, when calculating the optimal solution for the dynamic lot-sizing model with time windows, as implied in Property 2, the most important information for decomposition is the periods in which productions are established. Under the condition that γ is a production period for $\gamma = 1, 2, \ldots, T+1$, let $F(\gamma)$ be the optimal cost in satisfying the demands whose LDDs are strictly less than γ, i.e. the demands $\beta(i)$ for $1 \leq i \leq b(\gamma - 1)$. Note that no setup cost K_γ is incurred in period γ when calculating $F(\gamma)$ but only unit production cost p_γ. We consider the cost $F(T+1)$ in detail. Recall that the unit costs p_{T+1} and g_{T+1} are set to ∞. Hence, no demands will be replenished by the production in period $T+1$ but will be by productions in periods $1, 2, \ldots, T$. This means that $F(T+1)$ is the very optimal cost we would like to find. We can compute $F(\gamma)$ for $\gamma = 1, 2, \ldots, T+1$ by the following recursion procedure:

$$F(1) = 0$$
$$F(\gamma) = \min_{1 \le \lambda < \gamma} \left\{ F(\lambda) + K_\lambda + p_\lambda \cdot \text{phload}(\lambda, \gamma) + \text{hcost}(\lambda, \gamma) + \text{pgcost}(\lambda, \gamma) \right\}.$$

Given all the necessary cost data by preprocessing, the optimal solution can be found in $O(T^2)$. Now, the remaining thing for computing $F(T+1)$ is to find the cost data for intervals $[\lambda, \gamma]$ for $1 \le \lambda < \gamma \le T+1$, which will be provided in the next two sections.

4 Selection measures

Keeping in mind the definitions of the cost data hload(λ, γ), hcost(λ, γ), phload(λ, γ), and pgcost(λ, γ) for $1 \le \lambda < \gamma \le T+1$, we can compute them easily in an enumerative way, although inefficient. However, in order to obtain the cost data for $[\lambda, \gamma]$ with minimal computational burden, we may need to develop firstly a scheme that efficiently determines replenishment periods λ or γ for the demands with $[\lambda, \gamma]$ and secondly procedures to calculate hload(λ, γ), hcost(λ, γ), and phload(λ, γ) for the demands replenished in period λ, and a procedure to calculate pgcost(λ, γ) for the demands replenished in period γ. We first take into account how to efficiently determine the replenishment periods for the demands with $[\lambda, \gamma], 1 \le \lambda < \gamma \le T+1$. In the period $T+1$, the production and backlogging costs of $T+1$ are set to ∞. This makes it easy to obtain the cost data for the interval $[\lambda, T+1], 1 \le \lambda < T+1$ since the demands with the interval are all replenished by period λ. Hence, we will focus on the intervals $[\lambda, \gamma], 1 \le \lambda < \gamma \le T$.

Consider the demands i in the interval $[\lambda, \gamma]$, i.e., those i with $[E_i, L_i] \subseteq [\lambda, \gamma]$. Note that any 'proper' demand i with the interval $[\lambda, \gamma]$ satisfies that $[E_i, L_i] \subseteq [\lambda, \gamma]$. The decision whether a demand i in the interval $[\lambda, \gamma]$ is replenished either at period λ or γ can be easily made based on by the value of $(p_\lambda + h_{\lambda, E_i - 1}) - (p_\gamma + g_{L_i, \gamma - 1})$. If it is no larger than zero, then the demand must be replenished by production at period λ, otherwise it is replenished by production at period γ. Let $\theta_i(\lambda, \gamma) = (p_\lambda + h_{\lambda, E_i - 1}) - (p_\gamma + g_{L_i, \gamma - 1})$ and we call it *selection measure* for demand i in the interval $[\lambda, \gamma]$. In particular, for the interval $[1, T]$ we sort its demands by nondecreasing order of the selection measure $\theta_i(1, T)$ and let $\pi(i)$ be the resulting list of demands ordered so that $\theta_{\pi(1)}(1, T) \le \theta_{\pi(2)}(1, T) \le \cdots \le \theta_{\pi(n)}(1, T)$. Then the following property enables us to efficiently determine replenishment period either λ or γ for demands in the interval $[\lambda, \gamma]$ just by keeping the sorted list π.

Property 3 *For any two demands i and j in the interval* $[\lambda, \gamma]$, $1 \le \lambda < \gamma \le T$, *it holds that* $\theta_i(\lambda, \gamma) \le \theta_j(\lambda, \gamma)$ *if and only if* $\theta_i(1, T) \le \theta_j(1, T)$.

Proof Note that $h_{1, E_i - 1} = h_{1, \lambda - 1} + h_{\lambda, E_i - 1}$ if $\lambda \le E_i$ and $g_{L_i, T-1} = g_{L_i, \gamma - 1} + g_{\gamma, T-1}$ if $L_i \le \gamma$. Hence, if demand i is in the interval $[\lambda, \gamma](\lambda \le E_i \le L_i \le \gamma)$, we have

$$\begin{aligned} \theta_i(\lambda, \gamma) &= (p_\lambda + h_{\lambda, E_i - 1}) - (p_\gamma + g_{L_i, \gamma - 1}) \\ &= (p_1 + h_{1, E_i - 1}) - (p_T + g_{L_i, T-1}) + (p_\lambda - p_\gamma - p_1 + p_T) - h_{1, \lambda - 1} + g_{\gamma, T-1} \\ &= \theta_i(1, T) + (p_\lambda - p_\gamma - p_1 + p_T) - h_{1, \lambda - 1} + g_{\gamma, T-1}. \end{aligned}$$

We see that $\theta_i(\lambda, \gamma)$ is written by $\theta_i(1, T)$ and the constant $(p_\lambda + p_\gamma - p_1 - p_T)$ - $h_{1,\lambda-1} + g_{\gamma,T-1}$ which does not depend on the index i but only on the periods λ and γ. This implies that $\theta_i(\lambda, \gamma) \le \theta_j(\lambda, \gamma)$ if and only if $\theta_i(1, T) \le \theta_j(1, T)$, for any two demands i and j in $[\lambda, \gamma]$, thereby proving the property. □

Thus, as far as demands in $[\lambda, \gamma]$ are concerned, their order by selection measure $\theta_i(\lambda, \gamma)$ remains the same as that in the list π. This invariant property of the list π will play a crucial role in computing the cost data for 'proper' demands. However, it does not hold for λ-crossing demands because we can easily find cases that $\theta_i(\lambda, \gamma) < \theta_j(\lambda, \gamma)$ even though $\theta_i(1, T) > \theta_j(1, T)$ when demand i or j does not belong to $[\lambda, \gamma]$. Therefore we will later develop another selection measure for λ-crossing demands.

Using the selection measure $\theta_i(\lambda, \gamma)$, we can divide the list of demands into two disjoint lists: one containing demands with the value of selection measures at most zero and the other containing demands with the value of selection measures greater than zero. The proper demands in the former list will be replenished in period λ whereas the proper demands in the latter will be in period γ. Let $c(\lambda, \gamma)$ for $1 \le \lambda < \gamma \le T$ be the largest index $i \ge 0$ such that $j > i$ for all 'proper' demands $\pi(j)$ with $\theta_{\pi(j)}(\lambda, \gamma) > 0$. If no such index exists, we let $c(\lambda, \gamma) = n$. We call $c(\lambda, \gamma)$ the *critical index* for the interval $[\lambda, \gamma]$. Hence, according to this critical index, we can choose either the period λ or γ to replenish a proper demand.

For a proper demand i with $[\lambda - 1, \gamma]$, consider the value of $\theta_i(\lambda, \gamma) - \theta_i(\lambda - 1, \gamma)$:

$$\begin{aligned}
\theta_i(\lambda, \gamma) &- \theta_i(\lambda - 1, \gamma) \\
&= ((p_\lambda + h_{\lambda,E_i-1}) - (p_\gamma + g_{L_i,\gamma-1})) - ((p_{\lambda-1} + h_{\lambda-1,E_i-1}) - (p_\gamma + g_{L_i,\gamma-1})) \\
&= p_\lambda - p_{\lambda-1} - h_{\lambda-1}.
\end{aligned}$$

Since $p_\lambda \le p_{\lambda-1}$, we have $\theta_i(\lambda, \gamma) - \theta_i(\lambda - 1, \gamma) \le 0$, i.e.,

$$\theta_i(\lambda, \gamma) \le \theta_i(\lambda - 1, \gamma).$$

Note that this relationship between selection measures for the intervals $[\lambda - 1, \gamma]$ and $[\lambda, \gamma]$ holds for all proper demands with $[\lambda - 1, \gamma]$ and thus implies that for $2 \le \lambda < \gamma \le T$

$$c(\lambda - 1, \gamma) \le c(\lambda, \gamma).$$

By this inequality, we can quickly find the critical index for an interval by searching only a portion of the list π rather than the whole list. Let $c(\lambda, T + 1) = n$ and $c(0, \gamma) = 0$ for any $1 \le \lambda, \gamma \le T$. Then, given $c(\lambda - 1, \gamma)$, we can get the index $c(\lambda, \gamma)$ for $1 \le \lambda < \gamma \le T$ by the following:

$$c(\lambda, \gamma) = \min \begin{cases} \min\{i - 1 : \theta_{\pi(i)}(\lambda, \gamma) > 0, \lambda < E_{\pi(i)} \le L_{\pi(i)} < \gamma, c(\lambda - 1, \gamma) + 1 \le i \le n\}, \\ n, \end{cases} \quad (1)$$

Thus, for a period γ, we can find all the critical indices $c(\lambda, \gamma)$ for $1 \le \lambda < \gamma$ in time $O(\max\{T, n\})$ using the recursion (1). Hence, all the critical indices $c(\lambda, \gamma)$ for $1 \le \lambda < \gamma \le T$ can be obtained in time $O(\max\{T^2, nT\})$.

Analogously with $\theta_i(\lambda, \gamma)$, we define another selection measure $\theta'_i(\lambda, \gamma)$ which will be used for deciding whether λ-crossing demands i with the interval $[\lambda, \gamma]$ must be replenished in period λ or γ. Let $\theta'_i(\lambda, \gamma) = p_\lambda - (p_\gamma + g_{L_i,\gamma-1})$. Thus, we know that if $\theta'_i(\lambda, \gamma) \le 0$ then the demand i will be replenished in period λ otherwise in period γ. Similar to Property 3, we also have an invariance property for the selection measure

$\theta'_i(\lambda,\gamma)$, which can easily be shown. We notice that the property holds not only for the λ-crossing demands but for any demands with LDDs at most γ.

Property 4 *For any two demands i and j with $L_i \le \gamma$ and $L_j \le \gamma$, it holds that $\theta'_i(\lambda,\gamma) \le \theta'_j(\lambda,\gamma)$ if and only if $\theta'_i(1,T) \le \theta'_j(1,T)$ for $1 \le \lambda < \gamma \le T$.*

Let π' be the sorted list of demands by the measure $\theta'_i(1,T)$ for the interval $[1,T]$. The critical index $c'(\lambda,\gamma)$ for the λ-crossing demands with $[\lambda,\gamma]$, $1 \le \lambda < \gamma \le T$, is defined similarly as $c(\lambda,\gamma)$ and can be found by

$$c'(\lambda,\gamma) = \min \begin{cases} \min\{i-1 : \theta'_{\pi'(i)}(\lambda,\gamma) > 0, E_{\pi'(i)} \le \lambda \le L_{\pi'(i)} < \gamma, c'(\lambda-1,\gamma)+1 \le i \le n\}, \\ n \end{cases} \quad (2)$$

where $c'(\lambda, T+1) = n$ and $c'(0,\gamma) = 0$ for any $1 \le \lambda,\gamma \le T$. Thus, given a period γ, we can find all the critical indices $c'(\lambda,\gamma)$ for $1 \le \lambda < \gamma$ in time $O(\max\{T,n\})$ using the recursion (2).

5 Computing window cost data

In this section, we deal with how to compute the cost data for $[\lambda,\gamma]$, i.e., hload(λ,γ), hcost(λ,γ), phload(λ,γ), and pgcost(λ,γ), given the sorted lists of demands π and π'. They will be found by recursion from the cost data for $[\lambda-1,\gamma]$ and the basis cost data for $[1,\gamma]$ will be calculated in a simple enumerative way.

First, we would like to present recursion formulas for deriving the cost data for $[\lambda,\gamma]$ based on the cost data for $[\lambda-1,\gamma]$, $\lambda = 2,3,\ldots,\gamma$. The formulas are easily established once we know the relationship between the demands with $[\lambda-1,\gamma]$ and those with $[\lambda,\gamma]$. Taking into detailed account the demands with $[\lambda-1,\gamma]$, which consist of demands with LDD $= \lambda-1$ and demands with LDD $> \lambda-1$, then the demands with $[\lambda-1,\gamma]$ whose LDD $> \lambda-1$ are, in fact, the demands with $[\lambda,\gamma]$, which are further classified into the λ-crossing and proper demands. From now on, the demands with $[\lambda-1,\gamma]$ are thought to consist of demands with LDD $= \lambda-1$, λ-crossing and proper demands with $[\lambda,\gamma]$. Then, for the three sets of demands we observe eight types of demands based on the replenishment periods in the intervals $[\lambda-1,\gamma]$ and $[\lambda,\gamma]$ (See Table 3).

For the demands with $[\lambda-1,\gamma]$ whose LDD $= \lambda-1$, we have two types: type-B_1 in which the demands are replenished in period $\lambda-1$ and type-B_2 in which those are

Table 3 Types of demands with $[\lambda-1,\gamma]$

Demands with $[\lambda-1,\gamma]$	Types	Replenishment periods	
		$[\lambda-1,\gamma]$	$[\lambda,\gamma]$
Demands with $L_i = \lambda-1$	B_1	$\lambda-1$	–
	B_2	γ	–
λ-crossing demands with $[\lambda,\gamma]$	C_1	$\lambda-1$	λ
	C_2	γ	λ
	C_3	γ	γ
Proper demands with $[\lambda,\gamma]$	P_1	$\lambda-1$	λ
	P_2	γ	λ
	P_3	γ	γ

replenished in period γ. The demands of type-B_1 and of type-B_2 are mutually exclusive in the sense that they are all either exclusively type-B_1 or exclusively type-B_2 because all they have the same unit replenishment cost and hence they are replenished in the same period, either λ or γ.

In the λ-crossing demands with $[\lambda, \gamma]$, we have three types: type-C_1 in which the demands are replenished in period $\lambda - 1$ with respect to the interval $[\lambda - 1, \gamma]$ and in period λ with respect to $[\lambda, \gamma]$; type-C_2 in which they are replenished in period γ with $[\lambda\text{-}1, \gamma]$ but in λ with $[\lambda, \gamma]$; and type-C_3 in which they are replenished in period γ with respect to both the intervals $[\lambda\text{-}1, \gamma]$ and $[\lambda, \gamma]$. The reverse case of type-C_2 does not exist that it is economical to satisfy a λ-crossing demand by the production in period $\lambda - 1$ with respect to $[\lambda - 1, \gamma]$ while it is advantageous to satisfy the demand by the production in period γ with respect to $[\lambda, \gamma]$. This is because if the unit replenishment cost in period $\lambda - 1$ (i.e., $p_{\lambda-1} + h_{\lambda-1,E_i}$) is at most that in period γ (i.e., $p_\gamma + g_{L_i,\gamma-1}$), then the cost in period λ is also no greater than that in period γ due to the cost structure of $p_{t-1} \geq p_t$.

Likewise, we can also classify the proper demands with $[\lambda, \gamma]$ into three types, types-P_1, -P_2, and -P_3 corresponding to the types-C_1, -C_2, and -C_3, respectively. These eight types are summarized in Table 3.

Based on the eight types and the cost data $\text{hload}(\lambda-1, \gamma)$, $\text{hcost}(\lambda-1, \gamma)$, $\text{phload}(\lambda-1, \gamma)$, and $\text{pgcost}(\lambda-1, \gamma)$, we first present the methods for calculating $\text{hload}(\lambda, \gamma)$ and $\text{hcost}(\lambda, \gamma)$, followed by those for $\text{phload}(\lambda, \gamma)$ and $\text{pgcost}(\lambda, \gamma)$.

5.1 Computing hload(λ, γ) and hcost(λ, γ)

The data $\text{hload}(\lambda, \gamma)$ and $\text{hcost}(\lambda, \gamma)$ for $2 \leq \lambda < \gamma \leq T$ will be computed from $\text{hload}(\lambda - 1, \gamma)$, and $\text{hcost}(\lambda - 1, \gamma)$, respectively. To efficiently compute $\text{hload}(\lambda, \gamma)$ from $\text{hload}(\lambda - 1, \gamma)$, we first need to investigate in detail the demands which are used in computing the data $\text{hload}(\lambda - 1, \gamma)$. The demands associated with $\text{hload}(\lambda - 1, \gamma)$ are all proper with $[\lambda - 1, \gamma]$ and are replenished in period $\lambda - 1$. In terms of the eight types, they are characterized as type-C_1 with EDD$= \lambda$ and type-P_1. So, $\text{hload}(\lambda - 1, \gamma)$ can be written as:

$$\text{hload}(\lambda - 1, \gamma) = \left(\text{type-}C_1 \text{ with EDD} = \lambda\right) + \left(\text{type-}P_1\right).$$

In the case of $\text{hload}(\lambda, \gamma)$, the associated demands are the ones proper with $[\lambda, \gamma]$ and are replenished in period λ, i.e., those of types-P_1 and -P_2. Hence, with the representation of $\text{hload}(\lambda - 1, \gamma)$ above, we can denote $\text{hload}(\lambda, \gamma)$ as follows:

$$\begin{aligned}\text{hload}(\lambda, \gamma) &= \left(\text{type-}P_1\right) + \left(\text{type-}P_2\right)\\ &= \text{hload}(\lambda - 1, \gamma) - \left(\text{type-}C_1 \text{ with EDD} = \lambda\right) + \left(\text{type-}P_2\right).\end{aligned}$$

Now, we consider the demands of type-C_1 with EDD$= \lambda$ and type-P_2. Note that the demands with $[\lambda - 1, \gamma]$ whose EDD$= \lambda$ are $\alpha(i)$ with $L_{\alpha(i)} \leq \gamma - 1$ for $i = a(\lambda - 1) + 1, a(\lambda - 1) + 2, \ldots, a(\lambda)$. Moreover, among the demands with $[\lambda - 1, \gamma]$ whose EDD$= \lambda$, those replenished in period $\lambda - 1$ rather than in γ are $\alpha(i)$ for $i = a(\lambda - 1) + 1, a(\lambda - 1) + 2, \ldots, a(\lambda)$ satisfying the two conditions:

$$L_{\alpha(i)} \leq \gamma - 1 \quad \text{and} \quad p_{\lambda-1} + h_{\lambda-1} \leq p_\gamma + g_{L_{\alpha(i)},\gamma-1}.$$

Using the indicator function f, for these two conditions we obtain that

$$L_{\alpha(i)} \le \gamma - 1 \quad \text{if and only if} \quad f(L_{\alpha(i)}, \gamma - 1) = 1 \text{ and}$$
$$p_{\lambda-1} + h_{\lambda-1} \le p_\gamma + g_{L_{\alpha(i)},\gamma-1} \quad \text{if and only if}\, f(p_{\lambda-1} + h_{\lambda-1}, p_\gamma + g_{L_{\alpha(i)},\gamma-1}) = 1.$$

Hence, the term $\big(\text{type-}C_1 \text{ with EDD} = \lambda\big)$ is represented by

$$\big(\text{type-}C_1 \text{ with EDD} = \lambda\big) = \sum_{i=a(\lambda-1)+1}^{a(\lambda)} f(L_{\alpha(i)}, \gamma - 1) f(p_{\lambda-1} + h_{\lambda-1}, p_\gamma + g_{L_{\alpha(i)},\gamma-1}) \cdot d_{\alpha(i)}.$$

Next, consider the demands of type-P_2. Recall that any demand i satisfying $f(\lambda, E_i - 1) \cdot f(L_i, \gamma - 1) = 1$ is proper with $[\lambda, \gamma]$. Also, note that demands $\pi(i)$ proper with $[\lambda, \gamma]$, for $i = c(\lambda - 1, \gamma) + 1, c(\lambda - 1, \gamma) + 2, \ldots, c(\lambda, \gamma)$, are replenished in period γ with $[\lambda - 1, \gamma]$ but in period λ with $[\lambda, \gamma]$. Hence, the total sum of type-P_2 demands is

$$\big(\text{type-}P_2\big) = \sum_{i=c(\lambda-1,\gamma)+1}^{c(\lambda,\gamma)} f(\lambda, E_{\pi(i)} - 1) f(L_{\pi(i)}, \gamma - 1) \cdot d_{\pi(i)}.$$

Finally, combining the terms $\big(\text{type-}C_1 \text{ with EDD} = \lambda\big)$ and $\big(\text{type-}P_2\big)$ the $\text{hload}(\lambda, \gamma)$ can be computed by,

$$\begin{aligned}\text{hload}(\lambda, \gamma) = \text{hload}(\lambda - 1, \gamma) &- \sum_{i=a(\lambda-1)+1}^{a(\lambda)} f(L_{\alpha(i)}, \gamma - 1) f(p_{\lambda-1} + h_{\lambda-1}, p_\gamma + g_{L_{\alpha(i)},\gamma-1}) \cdot d_{\alpha(i)} \\ &+ \sum_{i=c(\lambda-1,\gamma)+1}^{c(\lambda,\gamma)} f(\lambda, E_{\pi(i)} - 1) f(L_{\pi(i)}, \gamma - 1) \cdot d_{\pi(i)}. \qquad (3)\end{aligned}$$

We next consider how to compute the cost data $\text{hcost}(\lambda, \gamma)$ by investigating the demands with their costs consisting in $\text{hcost}(\lambda - 1, \gamma)$. Similar to $\text{hload}(\lambda - 1, \gamma)$, the associated demands for $\text{hcost}(\lambda - 1, \gamma)$ are those of type-C_1 with EDD $= \lambda$ and type-P_1. Thus, $\text{hcost}(\lambda - 1, \gamma)$ can be written as

$$\begin{aligned}\text{hcost}(\lambda - 1, \gamma) &= \begin{pmatrix}\text{hcost of type-}C_1 \\ \text{with EDD} = \lambda\end{pmatrix} + \begin{pmatrix}\text{hcost of type-}P_1 \\ \text{in period } \lambda - 1\end{pmatrix} \\ &= h_{\lambda-1} \cdot \begin{pmatrix}\text{type-}C_1 \\ \text{with EDD} = \lambda\end{pmatrix} \\ &\quad + h_{\lambda-1} \cdot \big(\text{type-}P_1\big) + \begin{pmatrix}\text{hcost of type-}P_1 \\ \text{in period } \lambda\end{pmatrix}.\end{aligned}$$

Since the demands of type-C_1 with EDD $= \lambda$ and type-P_1 correspond to those of $\text{hload}(\lambda - 1, \gamma)$, we have

$$\text{hcost}(\lambda - 1, \gamma) = h_{\lambda-1} \cdot \text{hload}(\lambda - 1, \gamma) + \begin{pmatrix}\text{hcost of type-}P_1 \\ \text{in period } \lambda\end{pmatrix}.$$

Now, consider the cost $\text{hcost}(\lambda, \gamma)$. As with $\text{hload}(\lambda, \gamma)$, we need to deal with the demands of types-P_1 and -P_2 and then their inventory holding costs. Considering the formula for $\text{hcost}(\lambda - 1, \gamma)$, we have

$$\begin{aligned}\text{hcost}(\lambda,\gamma) &= \begin{pmatrix}\text{hcost of type-}P_1\\ \text{in period }\lambda\end{pmatrix} + \begin{pmatrix}\text{hcost of type-}P_2\\ \text{in period }\lambda\end{pmatrix}\\ &= \text{hcost}(\lambda-1,\gamma) - h_{\lambda-1}\cdot\text{hload}(\lambda-1,\gamma) + \begin{pmatrix}\text{hcost of type-}P_2\\ \text{in period }\lambda\end{pmatrix}.\end{aligned}$$

Also, by the same argument in characterizing the demands of type-P_2 for hload(λ,γ), we see that

$$\begin{pmatrix}\text{hcost of type-}P_2\\ \text{in period }\lambda\end{pmatrix} = \sum_{i=c(\lambda-1,\gamma)+1}^{c(\lambda,\gamma)} f(\lambda, E_{\pi(i)}-1)f(L_{\pi(i)},\gamma-1)\cdot h_{\lambda,E_{\pi(i)}-1}\cdot d_{\pi(i)}.$$

Thus, we can find hcost(λ,γ) by the following formula:

$$\begin{aligned}\text{hcost}(\lambda,\gamma) &= \mathit{hcost}(\lambda-1,\gamma) - h_{\lambda-1}\cdot\text{hload}(\lambda-1,\gamma)\\ &+ \sum_{i=c(\lambda-1,\gamma)+1}^{c(\lambda,\gamma)} f(\lambda, E_{\pi(i)}-1)f(L_{\pi(i)},\gamma-1)\cdot h_{\lambda,E_{\pi(i)}-1}\cdot d_{\pi(i)}.\end{aligned} \tag{4}$$

5.2 Computing phload(λ,γ) and pgcost(λ,γ)

The data phload(λ,γ) and pgcost(λ,γ) for $2 \le \lambda < \gamma \le T$ shall be computed in the same way as that presented in the previous subsection. We first consider the components of demands of phload$(\lambda-1,\gamma)$ and phload(λ,γ) and then present the method to obtain phload(λ,γ) from phload$(\lambda-1,\gamma)$. We recall that phload$(\lambda-1,\gamma)$ is the production and inventory holding costs for the λ-crossing and proper demands with $[\lambda-1,\gamma]$ replenished in period $\lambda-1$. Using the eight types in Table 3, the demands for phload$(\lambda-1,\gamma)$ are characterized to be types-B_1, -C_1, and -P_1. Hence, we denote phload$(\lambda-1,\gamma)$ by

$$\text{phload}(\lambda-1,\gamma) = \left(\text{type-}B_1\right) + \left(\text{type-}C_1\right) + \left(\text{type-}P_1\right).$$

We also see that the demands for phload(λ,γ) are types-C_1, -C_2, -P_1, and -P_2. Then, we obtain with the formula for phload$(\lambda-1,\gamma)$ that

$$\begin{aligned}\text{phload}(\lambda,\gamma) &= \left(\text{type-}C_1\right) + \left(\text{type-}C_2\right) + \left(\text{type-}P_1\right) + \left(\text{type-}P_2\right)\\ &= \text{phload}(\lambda-1,\gamma) - \left(\text{type-}B_1\right) + \left(\text{type-}C_2\right) + \left(\text{type-}P_2\right).\end{aligned}$$

Consider the terms $\left(\text{type-}B_1\right)$, $\left(\text{type-}C_2\right)$, and $\left(\text{type-}P_2\right)$. Each demand of type-$B_1$ has LDD $= \lambda-1$ and hence it is one of the demands $\beta(i)$ for $i = b(\lambda-2)+1, b(\lambda-2)+2, \ldots, b(\lambda-1)$. In addition, each demand of type-B_1 is replenished in period $\lambda-1$. We notice that any demand with LDD $= \lambda-1$ is replenished in period $\lambda-1$ rather than in γ if $p_{\lambda-1} \le p_\gamma + g_{\lambda-1,\gamma-1}$ or $f(p_{\lambda-1}, p_\gamma + g_{\lambda-1,\gamma-1}) = 1$. Hence, the term $\left(\text{type-}B_1\right)$ is stated as

$$\left(\text{type-}B_1\right) = \sum_{i=b(\lambda-2)+1}^{b(\lambda-1)} f(p_{\lambda-1}, p_\gamma + g_{\lambda-1,\gamma-1})\cdot d_{\beta(i)}.$$

The demands of type-C_2 are replenished in period γ with $[\lambda-1,\gamma]$ while they are replenished in period λ with $[\lambda,\gamma]$. Also, they are λ-crossing with $[\lambda,\gamma]$. Recall that a demand i is λ-crossing with $[\lambda,\gamma]$ iff $f(E_i,\lambda)f(\lambda,L_i)f(L_i,\gamma-1) = 1$. Then, using the same argument in characterizing the term $\left(\text{type-}P_2\right)$ as $\sum_{i=c(\lambda-1,\gamma)+1}^{c(\lambda,\gamma)} f(\lambda, E_{\pi(i)}-$

$1)f(L_{\pi(i)}, \gamma-1)\cdot d_{\pi(i)}$ in the previous subsection, we can see that the term $(\text{type-}C_2)$ is given by

$$(\text{type-}C_2) = \sum_{i=c'(\lambda-1,\gamma)+1}^{c'(\lambda,\gamma)} f(E_{\pi'(i)},\lambda)f(\lambda, L_{\pi'(i)})f(L_{\pi'(i)},\gamma-1)\cdot d_{\pi'(i)}.$$

Then, inserting the terms $(\text{type-}B_1)$, $(\text{type-}C_2)$, and $(\text{type-}P_2)$ into the formula for phload(λ,γ), we finally obtain that

$$\begin{aligned}\text{phload}(\lambda,\gamma) &= \text{phload}(\lambda-1,\gamma) - \sum_{i=b(\lambda-2)+1}^{b(\lambda-1)} f(p_{\lambda-1}, p_\gamma + g_{\lambda-1,\gamma-1})\cdot d_{\beta(i)}\\ &\quad + \sum_{i=c'(\lambda-1,\gamma)+1}^{c'(\lambda,\gamma)} f(E_{\pi'(i)},\lambda)f(\lambda, L_{\pi'(i)})f(L_{\pi'(i)},\gamma-1)\cdot d_{\pi'(i)}\\ &\quad + \sum_{i=c(\lambda-1,\gamma)+1}^{c(\lambda,\gamma)} f(\lambda, E_{\pi(i)}-1)f(L_{\pi(i)},\gamma-1)\cdot d_{\pi(i)}. \end{aligned} \tag{5}$$

Next, consider the cost pgcost(λ,γ) which is the total production and backlogging costs in period γ. As was done previously, we determine the demands for pgcost$(\lambda-1,\gamma)$ to be types-B_2, -C_2, -C_3, -P_2, and -P_3. Hence we have

$$\begin{aligned}\text{pgcost}(\lambda-1,\gamma) &= \begin{pmatrix}\text{pgcost of}\\ \text{type-}B_2\end{pmatrix} + \begin{pmatrix}\text{pgcost of}\\ \text{type-}C_2\end{pmatrix} + \begin{pmatrix}\text{pgcost of}\\ \text{type-}C_3\end{pmatrix}\\ &\quad + \begin{pmatrix}\text{pgcost of}\\ \text{type-}P_2\end{pmatrix} + \begin{pmatrix}\text{pgcost of}\\ \text{type-}P_3\end{pmatrix}.\end{aligned}$$

In the case of pgcost(λ,γ), its demands are types-C_3, and -P_3. Hence, from the formula for pgcost$(\lambda-1,\gamma)$ we have

$$\begin{aligned}\text{pgcost}(\lambda,\gamma) &= \begin{pmatrix}\text{pgcost of}\\ \text{type-}C_3\end{pmatrix} + \begin{pmatrix}\text{pgcost of}\\ \text{type-}P_3\end{pmatrix}\\ &= \text{pgcost}(\lambda-1,\gamma) - \begin{pmatrix}\text{pgcost of}\\ \text{type-}B_2\end{pmatrix} - \begin{pmatrix}\text{pgcost of}\\ \text{type-}C_2\end{pmatrix} - \begin{pmatrix}\text{pgcost of}\\ \text{type-}P_2\end{pmatrix}.\end{aligned}$$

Note that the demands of type-B_2 are exclusive with those of type-B_1. That is, demand $\beta(i)$ is type-B_2 if $b(\lambda-2)+1 \le i \le b(\lambda-1)$ and $p_{\lambda-1} > p_\gamma + g_{\lambda-1,\gamma-1}$ or $f(p_{\lambda-1}, p_\gamma + g_{\lambda-1,\gamma-1}) = 0$. Hence, the replenishment cost in period γ for demands of type-B_2 is given by

$$\begin{pmatrix}\text{pgcost of}\\ \text{type-}B_2\end{pmatrix} = \sum_{i=b(\lambda-2)+1}^{b(\lambda-1)} (1 - f(p_{\lambda-1}, p_\gamma + g_{\lambda-1,\gamma-1}))\cdot(p_\gamma + g_{\lambda-1,\gamma-1})\cdot d_{\beta(i)}.$$

The two terms $(\text{pgcost of type-}C_2)$ and $(\text{pgcost of type-}P_2)$ are easily calculated by multiplying the replenishment costs in the period γ to the terms $(\text{type-}C_2)$ and $(\text{type-}P_2)$. Hence, we have

$$\left(\text{pgcost of type-}C_2\right) = \sum_{i=c'(\lambda-1,\gamma)+1}^{c'(\lambda,\gamma)} f(E_{\pi'(i)},\lambda)f(\lambda,L_{\pi'(i)}) \times f(L_{\pi'(i)},\gamma-1)\cdot(p_\gamma + g_{L_{\pi'(i)},\gamma-1})\cdot d_{\pi'(i)},$$

$$\left(\text{pgcost of type-}P_2\right) = \sum_{i=c(\lambda-1,\gamma)+1}^{c(\lambda,\gamma)} f(\lambda,E_{\pi(i)}-1) \times f(L_{\pi(i)},\gamma-1)(p_\gamma + g_{L_{\pi(i)},\gamma-1})\cdot d_{\pi(i)}.$$

Then, the cost of $\text{pgcost}(\lambda,\gamma)$ is obtained by

$$\begin{aligned}\text{pgcost}(\lambda,\gamma) = {} & \text{pgcost}(\lambda-1,\gamma) \\ & - \sum_{i=b(\lambda-2)+1}^{b(\lambda-1)} (1-f(p_{\lambda-1},p_\gamma+g_{\lambda-1,\gamma-1}))\cdot(p_\gamma+g_{\lambda-1,\gamma-1})\cdot d_{\beta(i)} \\ & - \sum_{i=c'(\lambda-1,\gamma)+1}^{c'(\lambda,\gamma)} f(E_{\pi'(i)},\lambda)f(\lambda,L_{\pi'(i)}) \\ & \times f(L_{\pi'(i)},\gamma-1)\cdot(p_\gamma+g_{L_{\pi'(i)},\gamma-1})\cdot d_{\pi'(i)} \\ & - \sum_{i=c(\lambda-1,\gamma)+1}^{c(\lambda,\gamma)} f(\lambda,E_{\pi(i)}-1)f(L_{\pi(i)},\gamma-1)(p_\gamma+g_{L_{\pi(i)},\gamma-1})\cdot d_{\pi(i)}.\end{aligned} \tag{6}$$

Collecting the formulas (3) through (6), we can compute the cost data for $[\lambda,\gamma]$ from the data for $[\lambda-1,\gamma]$. This calculation is done by the procedure CostingByEDD (Fig. 2). Note that we can also use this procedure in calculating cost data for intervals $[\lambda, T+1], 1 \le \lambda < T+1$.

Then to provide the basis cost data, we present the procedure CostingByLDD, which computes in a simple manner all the cost data for $[1,\gamma]$, $\gamma = 2,3,\ldots,T+1$ (Fig. 3). By the definitions of $c(1,\gamma)$, $c'(1,\gamma)$ and the indicator function, it is clear that the formulations in the procedure generate the cost data for $[1,\gamma]$.

The final task remaining is to initialize the data with respect to the interval $[1,T]$ and to provide a systematic way to compute cost data for all intervals. These are done via the procedure called WindowCosting (Fig. 4). Note that the selection measures calculated in this procedure are only for the interval $[1,T]$ and the sorting steps are performed only four times. It is clear that the procedures CostingByLDD $(1,\gamma)$ for $\gamma = 2,3,\ldots,T+1$ run in $O(nT)$. By the definition of a and b, it holds that $a(\lambda-1) \le a(\lambda)$ and $b(\lambda-1) \le b(\lambda)$. Then, considering the relationships between the critical indices $c(\lambda-1,\gamma) \le c(\lambda,\gamma)$ and $c'(\lambda-1,\gamma) \le c'(\lambda,\gamma)$, we can see that the pro-

CostingByEDD(λ, γ)

Find the critical indices $c(\lambda,\gamma)$, $c'(\lambda,\gamma)$ using the recursions (1) and (2), respectively;

Find *hload*(λ, γ), *hcost*(λ, γ), *phload*(λ, γ), and *pgcost*(λ, γ) using (3), (4), (5) and (6), respectively;

Fig. 2 Procedure CostingByEDD

CostingByLDD(1, γ)

Find the critical indices $c(1,\gamma)$, $c'(1,\gamma)$ using the recursions (1) and (2), respectively;

$$hload(1,\gamma) = \sum_{i=1}^{c(1,\gamma)} f(1, E_{\pi(i)} - 1) f(L_{\pi(i)}, \gamma - 1) \cdot d_{\pi(i)};$$

$$hcost(1,\gamma) = \sum_{i=1}^{c(1,\gamma)} f(1, E_{\pi(i)} - 1) f(L_{\pi(i)}, \gamma - 1) \cdot h_{1,E_{\pi(i)}-1} \cdot d_{\pi(i)};$$

$$phload(1,\gamma) = \sum_{i=1}^{c'(1,\gamma)} f(E_{\pi'(i)}, 1) f(1, L_{\pi'(i)}) f(L_{\pi'(i)}, \gamma - 1) \cdot d_{\pi'(i)} + \sum_{i=1}^{c(1,\gamma)} f(1, E_{\pi(i)} - 1) f(L_{\pi(i)}, \gamma - 1) \cdot d_{\pi(i)};$$

$$pgcost(1,\gamma) = \sum_{i=c'(1,\gamma)+1}^{n} f(E_{\pi'(i)}, 1) f(1, L_{\pi'(i)}) f(L_{\pi'(i)}, \gamma - 1) \cdot (p_\gamma + g_{L_{\pi'(i)},\gamma-1}) \cdot d_{\pi'(i)} + \sum_{i=c(1,\gamma)+1}^{n} f(1, E_{\pi(i)} - 1) f(L_{\pi(i)}, \gamma - 1)(p_\gamma + g_{L_{\pi(i)},\gamma-1}) \cdot d_{\pi(i)};$$

Fig. 3 Procedure CostingByLDD

WindowCosting

Compute $\theta_i(1,T)$ and $\theta_i'(1,T)$ for $i = 1, 2, \ldots, n$;

Sort demands by the selection measures $\theta_i(1,T)$ and $\theta_i'(1,T)$, keeping the results to π and π', respectively;

Sort demands by EDD and LDD and keep the results to α with a and β with b, respectively;

for $\gamma = 2$ **to** T+1

  CostingByLDD(1, γ);

  for $\lambda = 2$ **to** $\gamma - 1$

    CostingByEDD(λ, γ);

Fig. 4 Procedure WindowCosting

cedures CostingByEDD(λ, γ) for $\lambda = 2, 3, \ldots, \gamma - 1$ are performed in $O(\max\{T, n\})$. Hence, the overall complexity of the procedure WindowCosting is $O(\max\{T^2, nT\})$.

6 Concluding remarks

For the dynamic lot-sizing model with demand time windows where backlogging is allowed, we have designed an optimal algorithm with complexity $O(\max\{T^2, nT\})$. This algorithm is mainly based on the order invariance property in which the order of demands, according to the difference between the unit cost when replenished from period 1 and that from period T, is invariant for any two consecutive production periods λ and γ whenever the demands belong to the interval $[\lambda, \gamma]$. The preprocessing procedure WindowCosting with $O(\max\{T^2, nT\})$ is used to compute the cost data with respect to intervals of periods. Once the cost data are found, the main dynamic programming procedure is performed in $O(T^2)$. As we consider it very difficult to create a better preprocessing algorithm than WindowCosting, the development of a more advanced optimal solution procedure with better complexity than $O(\max\{T^2, nT\})$ is unlikely.

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